PMM U.S.S.R., Vol. 46, No. 6, pp. 741-745, 1983.
Printed in Great Britain.

0021-8928/83 $7.50+0.00


UDC 531.36:534

# STOCHASTIC AUTO-OSCILLATIONS OF A NONLINEAR OSCILLATOR WITH IMPACT ENERGY ABSORBER*

T.Iu. DRUZHILOVSKAIA and Iu.I. NEIMARK

A nonlinear Van der Pol oscillator with impact energy absorber is considered. Dependence of all possible forms of the system phase pattern on parameters is determined. Conditions of steady stochastic motion onset are defined. The parameter space is divided in regions that qualitatively correspond to different phase patterns of the system.

**1. The investigated system.** The nonlinear Van der Pol oscillator

$$x^{\cdot\cdot} + 2\delta(1 - \alpha x^2)x^{\cdot} + \omega^2 x = 0 \qquad (1.1)$$

is the simplest typical system in which auto-oscillations are induced by the supply of energy at small oscillations and by its dissipation at large oscillations /1,2/. This simple physical concept may be refined by considering consecutive values of velocity $x^{\cdot}$ as the coordinate $x$ approaches zero, i.e. by investigating the point mapping of semiaxis $x = 0, x^{\cdot} > 0$ into itself, which is generated by phase trajectories of system (1.1). Curves of such point mappings appear in Fig.1, where curves 1 and 2 correspond to $\delta = -0.2$ and $\delta = -0.05$ for $\alpha = 1$, and curves 3 and 4 relate to $\delta = 0.05$ and $\delta = 0.2$ when $\alpha = -1$. These curves imply that in the case of $\delta < 0$ and $\alpha > 0$ and small $x$ the successive values of $x^{\cdot}$ are higher than the preceding ones, while for large $x$ they are lower. Owing to this dependence of the value of successive $x^{\cdot}$ on the preceding one, there exists an $x^{\cdot}$ whose value is the same as that of the following one, and corresponds to stable periodic auto-oscillations.

Introduction in the nonlinear oscillator (1.1) of energy absorption consequenct to a shock at $x = 0$ and $x^{\cdot} > a$ that reduces velocity $x^{\cdot}$ by some quantity $p$ is investigated below. It is shown that then not only periodic, but also stochastic auto-oscillations (one of the later cases is shown in Fig.2) may be generated. Depending on initial conditions and parameters, besides the usual periodic and stochastic auto-oscillations there may exist either only periodic or only stochastic auto-oscillations. A change of parameters may alter periodic auto-oscillations to stochastic, and also the emergence of stochastic auto-oscillations in consequence of loss of stability of the equilibrium state.

The indicated energy absorption mechanism thus reduces the nonlinear oscillator (1.1) to a system in which stochastic auto-oscillations are realized. That system is not only simple but, also, typical, since the appearance in it of stochastic auto-oscillations does not depend on the specific form of the absorber, but on its presence /3/. We would stress that the described shock mechanism is to be considered as an energy absorber only when the velocity $x_+^{\cdot}$

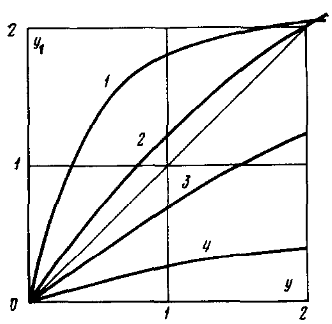



Fig.1

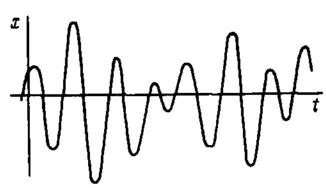



Fig.2

*Prikl.Matem.Mekhan.,46,No.6,pp.924-930,1982 after the shock is lower than $x_-^{\cdot}$ before the shock, which is evidently the case when $p < a$.

Investigation of the nonlinear oscillator (1.1) with additional energy absorber in the particular case of $\alpha = 0$ appears in /4/. Below, we consider the general case. All possible forms of point mapping of the secant $x = 0, x^{\cdot} > a$ into itself are determined and their dependence on parameters is established. The analysis is based on qualitative as well as on analytic considerations and, also, on numerical calculation on a computer. As a preliminary, we consider the nonlinear Van der Pol oscillator.

## 2. Point mapping generated by phase trajectories of Van der Pol equations.

In new variables

$$t_1 = t\omega^{-1}, \quad x_1 = |\alpha|^{1/2} x$$

the equations of motion of this system, with retained previous notation, are of the form

$$x^{\cdot\cdot} + 2\delta (1 - \alpha x^2) x^{\cdot} + x = 0 \ (x \neq 0 \ \text{ and } \ x = 0, \ x^{\cdot} < a) \qquad (2.1)$$
$$x_+^{\cdot} = x_-^{\cdot} - p \qquad (x = 0, \ x^{\cdot} \geqslant a)$$

where $x_+^{\cdot}$ and $x_-^{\cdot}$ are limit values of $x^{\cdot}$ for the increase and decrease of time up to $x = 0$ and $x^{\cdot} \geqslant a, \alpha = \pm 1$ depending on the sign of $\alpha$ in the input equation (1.1). The transformed system of equations contains three parameters $\delta, p, a$, when $p = 0$ it becomes the Van der Pol oscillator whose motions depend on parameter $\delta$ and the value of $\alpha = \pm 1$. The point mapping

$$y_1 = f(y) \qquad (2.2)$$

of the secant half-line $x = 0, y = x^{\cdot} > a$ is obtained numerically using a computer. The results are presented in Fig.1 in the form of curves, and tabulated below. Note that the substitution of $-\delta$ for $\delta$ reverses mapping (2.2)

Table 1

| $\delta$ | $y = 0{,}1$ | 0,5 | 1,0 | 1,5 | 2,0 | 2,5 | 3,0 | 3,5 | 4,0 |
|---|---|---|---|---|---|---|---|---|---|
| —0,01 | 0,11 | 0,26 | 1,05 | 1,54 | 2,00 | 2.42 | 2,80 | 3.14 | 3,44 |
| —0,1 | 0,19 | 0,06 | 1,47 | 1,81 | 2,01 | 2,12 | 2,18 | 2,22 | 2,25 |
| —0,3 | 0,68 | 0.01 | 2,00 | 2.05 | 2.06 | 2,07 | 2.07 | 2,07 | 2,08 |
| —0,5 | 1,79 | 1,78 | 2,17 | 2,17 | 2,17 | 2,17 | 2,17 | 2,17 | 2.17 |
| 0,01 | 0,09 | 2,14 | 0,93 | 1,36 | 1,78 | 2,16 | 2,50 | 2.82 | 3.10 |
| 0,1 | 0,05 | 0,47 | 0,49 | 0,67 | 0,81 | 0,90 | 0,97 | 1.02 | 1,05 |
| 0,3 | 0,01 | 0,87 | 0,12 | 0,16 | 0,19 | 0.20 | 0.21 | 0.22 | 0.22 |
| 0,5 | 0,003 | 0,53 | 0.02 | 0,03 | 0.03 | 0,04 | 0.04 | 0.04 | 0,04 |

When $\alpha = +1$ and $\delta < 0$ the point mapping curves intersect the bisectrix and correspond to the phase pattern of Fig.3,a. When $\alpha = +1$ and $\delta > 0$ the point mapping curves are obtained from the respective curves of Fig.1 by interchanging axes $y$ and $y_1$. A change of the equilibrium state stability and of the limit cycle to the opposite to that in Fig.3,a corresponds to this in the phase pattern. When in the case $\alpha = -1$ and $\delta > 0$ the point mapping curves lie entirely below the bisectrix and conform to the phase pattern of Fig.3,b. Finally, when $\alpha = -1$ and $\delta < 0$ this curve lie above the bisectrix. The presence of a single unstable equilibrium state corresponds to them in the phase pattern. The values of $y_1$ appearing in the upper part of the Table 1 correspond to $\alpha = 1$ and various values of $y$ and $\delta < 0$, while those in its lower part correspond to $\alpha = -1$ and various values of $y$ and $\delta > 0$.

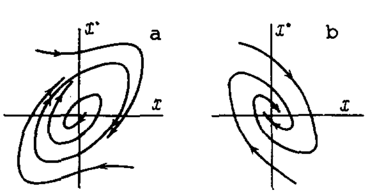


Fig.3

## 3. Investigation of point mapping when $p \neq 0$.

In this case the point mapping of segment $x = 0, x^{\cdot} \geqslant a$ into itself can be expressed in the form

$$y_1 = \begin{cases} f(y), & f(y) < a \\ f(y) - p, & f(y) \geqslant a \end{cases} \qquad (3.1)$$

where $f(y)$ is the function that defines the point mapping (2.2) generated by phase trajectories of the Van der Pol oscillator for the same parameters $\delta$ and $\alpha$. The form of this point mapping was explained in Sect.2. Various possible forms of curves of point mapping (3.1) shown in Figs.4—7 depend on four parameters $\delta, \alpha = \pm 1, p \geqslant 0$ and $a > 0$. The correspond, respectively, to the following cases: 1) $\alpha = 1$ and $\delta < 0$, 2) $\alpha = 1$ and $\delta > 0$, 3) $\alpha = -1$ and $\delta < 0$, and 4) $\alpha = -1$ and $\delta > 0$. Let us briefly describe these cases.

1. $\alpha = 1, \delta < 0$. Let $y_*$ and $y_0$ be points at which $f(y_*) = y_*$ and $f_y'(y_0) = 1$ $(y_0 < y_*)$. When a) $a > y_*$ mapping (3.1) contains an entirely stable stationary point $y_*$ which is asymptotically approached by all points, except point $y = 0$, in successive transformations (Fig. 4,a). When b) $a < y_*$ and $p < a - (f_y'(a))^{-1}$ mapping (3.1) has also the entirely stable point

$y_{**}$ different from point $y_*$ (Fig.4,b). When $a < y_*$ and $p > a - (f_y'(a))^{-1}$ several different cases presented in Figs.4,c–g, are possible. They are realized when the inequalities

c) $f(y_0) < a < y_0,\ a - (f_y'(a))^{-1} < p < a;$
d) $y_0 < a < f(y_0),\ a - (f_y'(a))^{-1} < p < f(y_0) - y_0;$
e) $y_0 < a < f(y_0),\ f(y_0) - y_0 < p < a;$
f) $0 < a < y_0,\ a - (f_y'(a))^{-1} < p < f(y_0) - y_0;$
g) $0 < a < y_0,\ f(y_0) - y_0 < p < a.$

are satisfied.

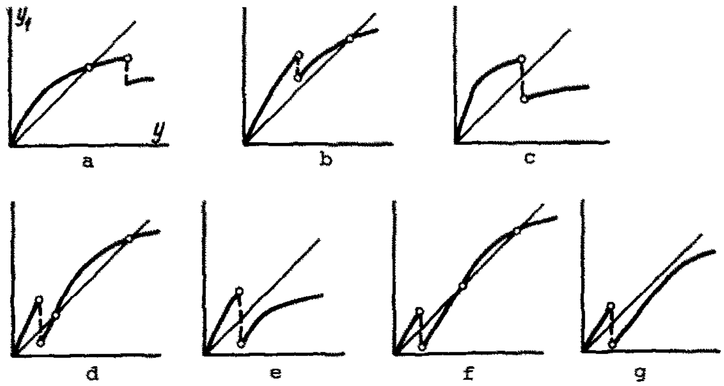


Fig.4

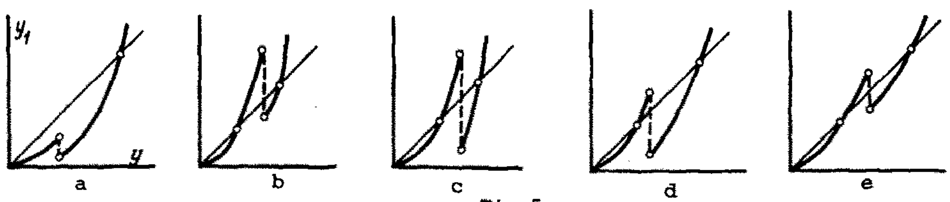
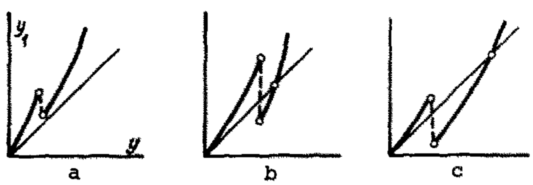


Fig.5



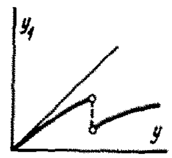


Fig.6 Fig.7

In case $e$ there is a stable stationary point $y_{**}$ and segment $J = (a - p, a)$ of stochastic motions, and in case g we have only the segment $J$ of stochastic motions. Thus in case g there exists an entirely stable stochastic auto-oscillation, and in case $e$ we have, besides the stable stochastic auto-oscillation, the usual stable periodic oscillation.

Let us clarify the concept of stochastic auto-oscillations. The curves of Figs.4, $e$ and g show that all points of segment $(0, y_{**})$ in case $e$, or of the whole semiaxis $(0, \infty)$ in case g pass after a certain number of transformations on segment $J = (a - p, a)$ and remain on the later. Since along the whole of segment $J$ the derivative $dy_1/dy > 1$, no stable periodic motions are possible along it. As shown in /5/, in that case Liapunov unstable stochastic motions, stochastic auto-oscillations, take place along segment $J$, which are exponentailly unstable and densely intersect segment $J$ (except some individual motions of general measure zero, among which there are various unstable periodic motions).

2. $\alpha = 1, \delta > 0$. This case admits five different types of point mapping. They are shown in Figs.5,a–e and correspond to the following range of parameter values:

a) $a < y_*,\ 0 < p < a;$
b) $a < y_*,\ p < \min(f(a) - a,\ a - y_*);$
c) $a > y_*,\ a - y < p < f(a) - a;$

d) $a > y_*$, $p > \max(f(a) - a, a - y_*)$;
e) $a > y_*$, $f(a) - a < p < a - y_*$

Curves of these mappings show that we have in all cases a stable equilibrium state with a limited attraction region, and that in some cases stochastic auto-oscillations also with a limited attraction region are possible.

3. $\alpha = -1$, $\delta < 0$. Possible forms of point mapping are shown in Figs.6,a–c which correspond to parameters that satisfy the conditions

a) $0 < p < a - (f_y'((a))^{-1}$;
b) $a - (f_y'(a))^{-1} < p < f(a) - a$;
c) $f(a) - a < p < a$

Note case c which corresponds to the presence of stochastic auto-oscillations with attraction region $(0, y_*)$, where $y_*$ (and also point $y = 0$ ) is an unstable stationary point.

4. $\alpha = -1$, $\delta > 0$. In this case independently of $a$ and $p < a$, we have one and the same form of point mapping shown in Fig.7. It corresponds to the entirely stable equilibrium state.

**4. Possible motion patterns and their corresponding regions of parameter values.** Above we described all possible forms of straight line transformation into itself, generated by phase trajectories of the nonlinear Van der Pols oscillator with shocks imparting the counter momentum $p < a$ at the instant of the oscillator passing throught the equilibrium state at a velocity higher than $a$. It was shown that a fairly large number of cases corresponding to various numbers and patterns of stable motions, including those of stable equilibrium and periodic and stochastic auto-oscillations, are possible.

The constraint $p < a$ is not imposed on the momentum of $p$ in the full investigation of the considered here mathematical model. This means that the effect of impact on the oscillator energy the latter can not only diminish but, also, be rised. Such widening of possible values of the momentum of $p$ increases the eight possible forms of phase patterns, shown in Fig. 4–7, by further three. Each phase pattern is defined by its stable motion clearly shown by the curves in Figs.4–7, except the regions of "suspected" stochastic motions, although the sufficient condition of stochasticity $dy_1/dy > 1$ is not satisfied in them. The case shown in Fig.4,e is one of these. Stochastic auto-oscillations, as well as stable periodic motions in which a phase point circulates several times around the equilibrium point, are also possible. Elucidation of this point involves a considerable amount of numerical calculations which, owing to the possibly fine structure of parameter space subdivision, are not always exhaustive. Determination of the phase pattern and of its dependence on parameters is in the remaining cases considerably simpler, in spite of being also based on the numerically determined function $f(y)$.

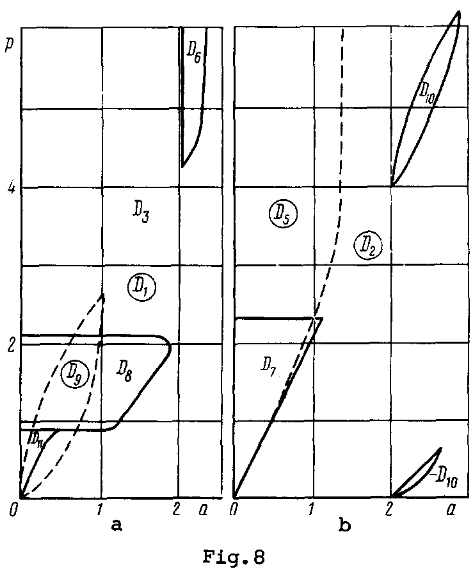


Fig.8

Subdivision of the parameter plane $a, p$ in regions of different phase pattern is shown in Figs. 8,a and b, respectively, for $\alpha = -1$ and $\delta = -0.1$ $(a)$ and for $\alpha = -1$ and $\delta = 0.1$ b by dash lines, and the regions denoted by encircled letters are shown in these figures for $\alpha = 1$ and $\delta = -0.2$ $(a)$, and for $\alpha = 1$ and $\delta = 0.03$ b , respectively and delineated by solid lines.

Regions $D_1, D_2, \ldots, D_{11}$ of parameter planes $a, p$ correspond to different phase patterns. To simplify their definition we use the following notation: $O^+$ and $O^-$ for stable and unstable equilibrium states, $\infty^+$ and $\infty^-$ for stability and instability at infinity, $\Gamma^+$ and $\Gamma^-$ for stable and unstable periodic motions, and $I$ for the stochastic auto-oscillation. Each of the $D_i$ regions can now be defined by a set of such symbols, viz.

$D_1(O^-, \infty^+)$, $D_2(O^+, \infty^-)$, $D_3(O^-, \Gamma^+, \infty^-)$
$D_4(O^+, \Gamma^-, \infty^+)$, $D_5(O^+, \Gamma^-, \Gamma^+, \infty^-)$, $D_6(O^-, \Gamma_1^+, \Gamma^-, \Gamma_2^+, \infty^-)$
$D_7(O^+, \Gamma_1^-, \Gamma_2^+, \infty^+)$, $D_8(O^-, I, \infty^-)$, $D_9(O^-, I, \Gamma^-, \infty^+)$
$D_{10}(O^+, \Gamma_1^-, I, \Gamma_2^-, \infty^+)$, $D_{11}(O^-, I, \Gamma^-, \Gamma^+, \infty^-)$

Thus region $D_1$ corresponds to the case of unlimited growth of oscillations, $D_2$ to the entire stability of the equilibrium state, $D_8$ to entirely stable stochastic motions, $D_3$ to entirely stable periodic auto-oscullations, and $D_{11}$ to the possibility of periodic and stochastic auto-oscillations, etc., depending on initial conditions. Stable motions and phase patterns undergo a change when passing through the boundaries of regions $D_i$.

REFERENCES


1. ANDRONOV A.A., VITT A.A., and KHAIKIN S.E., Theory of Oscillations. Pergamon Press, Book No.09981, 1966.
2. HAYASHI T., Nonlinear Oscillations in Physical Systems /Russian translation/. Moscow, MIR, 1968.
3. NEIMARK Iu.I., Physical mechanisms of stochastic oscillation self-generation. In: System Dynamics. Stability of Dynamic Systems and Control Processes. Izd. Gor'k.Univ., 1979.
4. NEIMARK Iu.I., Dynamic Systems and Controllable Processes. Moscow, NAUKA, 1978.
5. BUTENIN Iu.V., NEIMARK Iu.I. and FUFAEV N.A., Introduction to the Theory of Nonlinear Oscillations. Moscow, NAUKA, 1976.


Translated by J.J.D.